"Parking functions, the Shi arrangement and Macdonald polynomials" Philadelphia talk 03.05.2016 Arun Ram ①

Philadelphia Area Combinatorics and Alg Geometry Seminar

A parking function is $f:\{1,\ldots,n\}\to\mathbb{Z}_{\geq 0}$ such that

$(f(1),\ldots,f(n))$ arranged in ~~de~~ increasing order,

$(a_1,\ldots,a_n)$ satisfies $a_i\leq i-1$.

A generalization: $m=kn\pm 1$.

A $\frac{m}{n}$-parking function $f:\{1,\ldots,n\}\to\mathbb{Z}_{\geq 0}$ such that

$(f(1),\ldots,f(n))$ arranged in increasing order

$(a_1,\ldots,a_n)$ satisfies $a_i\leq mi-1$.

k=1 case:

If $n=2$: $(0,0)$, $(0,1)$, $(1,0)$ 3 total

If $n=3$: $(0,0,0)$ $(0,0,1)$ $(0,1,1)$ $(0,0,2)$ $(0,1,2)$
$(1,0,0)$ $(1,0,1)$ $(0,2,0)$ $(1,0,2)$ 16 total
$(0,1,0)$ $(1,1,0)$ $(2,0,0)$ $(0,2,1)$
$(2,0,1)$
$(1,2,0)$
$(2,1,0)$

CAGE Seminar Philadelphia talk 03.05.2016 (1)

Type GL_n For $1 \le i < j \le n$ let A. Ram

$$H^{\varepsilon_i - \varepsilon_j} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$$

$$\mathcal{A}^0 = \{H^{\varepsilon_i - \varepsilon_j} \mid 1 \le i < j \le n\}$$

$$H^{-(\varepsilon_i - \varepsilon_j) + k\delta} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i - x_j = k\}$$

$$\mathcal{A}^k = \{H^{-(\varepsilon_i - \varepsilon_j) + k\delta} \mid 1 \le i < j \le n\}.$$

Type G For $\alpha \in R^+$ let

$$H^\alpha = \{x \in \mathfrak{a}_\mathbb{R}^* \mid \alpha(x) = 0\}$$

$$\mathcal{A}^0 = \{H^\alpha \mid \alpha \in R^+\}$$

$$H^{-\alpha + k\delta} = \{x \in \mathfrak{a}_\mathbb{R}^* \mid \alpha(x) = k\}$$

$$\mathcal{A}^k = \{H^{\alpha + k\delta} \mid \alpha \in R^+\}$$

The *braid arrangement* is $\mathcal{A}^0$

The *Shi arrangement* is $\mathcal{A}^0 \cup \mathcal{A}^1 = \mathcal{A}^{[0,1]}$.

The *affine arrangement* is $\mathcal{A}^\mathbb{Z} = \bigsqcup_{k \in \mathbb{Z}} \mathcal{A}^k$

The *k-Shi arrangement* is $\mathcal{A}^{-k-1} \cup \cdots \cup \mathcal{A}^1 \cup \mathcal{A}^0 \cup \cdots \cup \mathcal{A}^k$.

The *finite Weyl group* is $W_0 = \{\text{conn. comps of } \mathfrak{a}_\mathbb{R} \setminus \mathcal{A}^0\}$

The *affine Weyl group* is $W_\mathbb{Z} = \{\text{conn. comps of } \mathfrak{a}_\mathbb{R} \setminus \mathcal{A}^\mathbb{Z}\}$.

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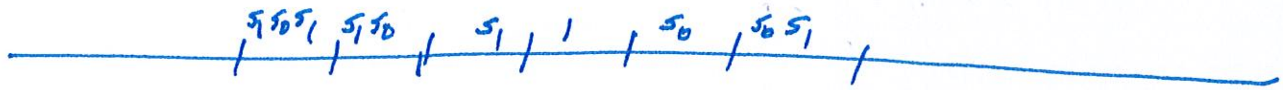
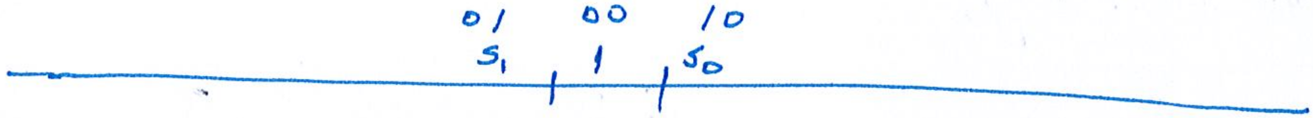
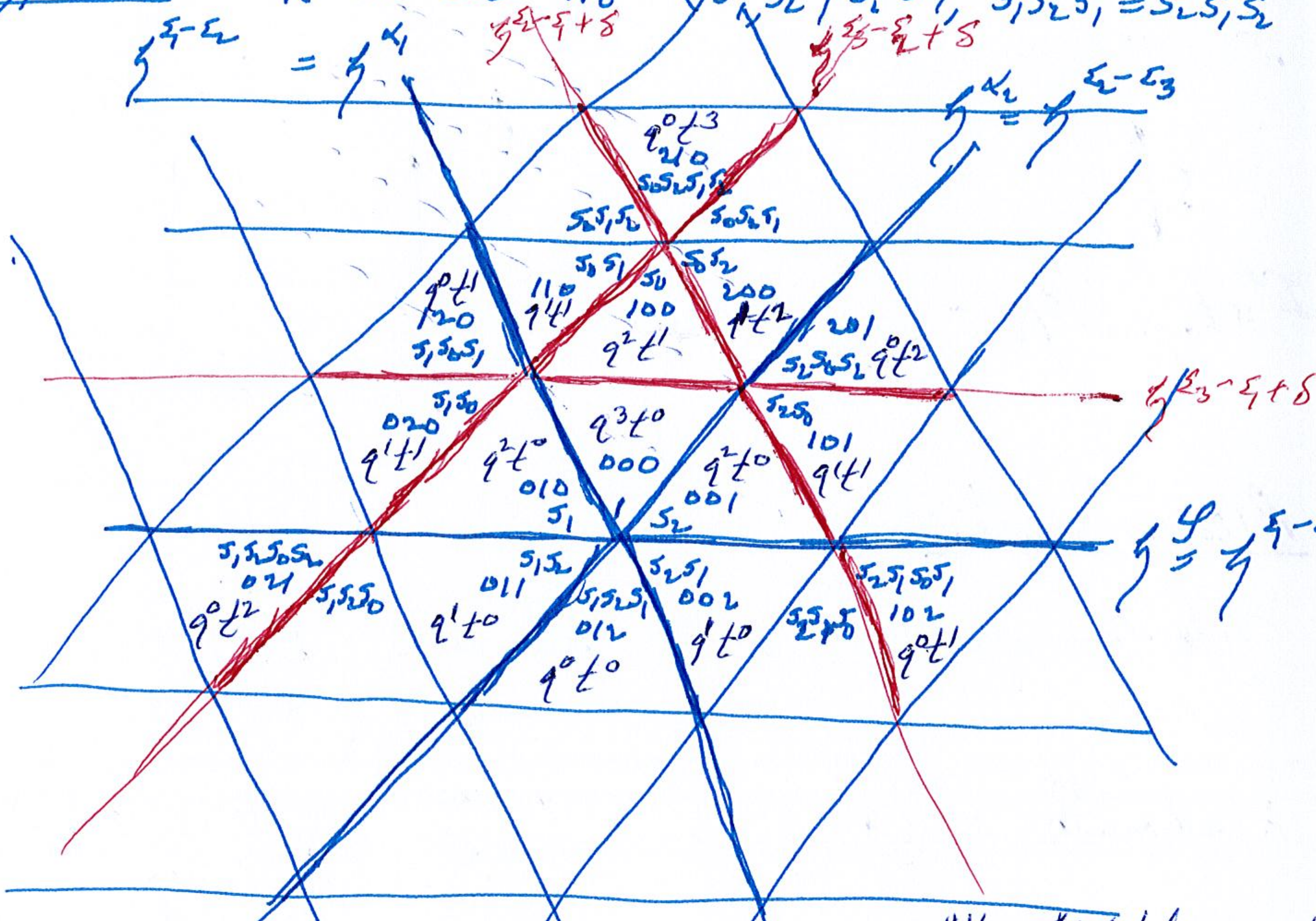
Type SL_2 $\quad \mathfrak{a}_{\mathbb{R}} = \mathbb{R}$ and $W_0 = \{1, s_1\}$ with $s_1^2 = 1$





Type SL_3 $\quad \mathfrak{a}_{\mathbb{R}} = \mathbb{R}^2$ and $W_0 = \langle s_1, s_2 \mid s_i^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$



$q^{\#\text{blue hyp between } w \text{ and } s_1s_2s_1}$

$t^{\text{diag in dom. chamber}}$

The map $\quad W \longrightarrow W^{-1}$ is a bijection

$w \longmapsto w^{-1}$

The map $\quad \{\text{regions of } \mathcal{A}^{\mathbb{Z}}\} \longrightarrow \{\text{regions of } \mathcal{A}^{\mathbb{Z}}\}$ is a bijection

$w \longmapsto w^{-1}$

Irreducible DAHA modules

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The <u>DAHA</u> is generated by

$$\tau_0, \tau_1, \ldots, \tau_n \quad \text{and} \quad y^{\varepsilon_1}, \ldots, y^{\varepsilon_n}$$

with relations

$$\sim\!\!\sim\!\!\sim\!\!\sim .$$

The <u>nonsymmetric Macdonald polynomials</u>

$\{E_w \mid w \in W_1\}$ form a basis of $L_{1+\frac{1}{n}}(\mathrm{triv})$, the irreducible finite dimensional DAHA module.

$$\begin{tikzcd}
 & & & E_{000} \arrow[d, "\tau_0"] & & & \\
 & & & E_{210} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & & & \\
 & & E_{120} \arrow[d, "\tau_2"'] & & E_{201} \arrow[d, "\tau_1"] & & \\
 & & E_{102} \arrow[dl, "\tau_0"'] \arrow[dr, "\tau_1"] & & E_{021} \arrow[dl, "\tau_2"] \arrow[dr, "\tau_0"] & & \\
 & E_{300} \arrow[d, "\tau_1"'] & & E_{012} \arrow[d, "\tau_0"'] & & E_{330} \arrow[d, "\tau_2"] & \\
 & E_{030} \arrow[dl, "\tau_2"'] \arrow[dr, "\tau_0"] & & E_{420} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & & E_{303} \arrow[dl, "\tau_0"'] \arrow[dr, "\tau_1"] & \\
E_{003} & & E_{240} & & E_{402} & & E_{033}
\end{tikzcd}$$

$$y^{\varepsilon_1} E_{240} = q^2 E_{240}$$
$$y^{\varepsilon_2} E_{240} = q^4 E_{240}$$
$$y^{\varepsilon_3} E_{240} = q^0 E_{240}$$

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Affine Springer Fibers

$$G = GL_n(\mathbb{C}[t,t^{-1}]) \quad \text{and} \quad I = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \middle| \begin{array}{l} a_i \in \mathbb{C}[t] \\ a_i(0) \in \mathbb{C}^\times \\ b_{ij} \in \mathbb{C}[t] \\ c_{ij} \in t\mathbb{C}[t] \end{array} \right\}$$

G/I is the affine flag variety and

$$G = \bigsqcup_{w \in W_{\mathbb{Z}}} IwI \qquad \text{(Bruhat decomposition).}$$

$G = GL_n(\mathbb{C}[t,t^{-1}])$ acts on $M_n(\mathbb{C}[t,t^{-1}])$ by conjugation.

$$\mathfrak{b} = \left\{ \begin{pmatrix} a_1 & & b_{ij} \\ & \ddots & \\ c_{ij} & & a_n \end{pmatrix} \middle| \begin{array}{l} a_i, b_{ij} \in \mathbb{C}[t] \\ c_{ij} \in t\mathbb{C}[t] \end{array} \right\}$$

Let

$$v = t^m \begin{pmatrix} \overbrace{0 \; \cdots \; 1}^{b} & & \\ & \ddots & 1 \\ t & \ddots & \\ \underbrace{\cdots \; t}_{b} & & 0 \end{pmatrix}.$$

The affine Springer fiber is

$$\mathcal{B}_{\frac{mn+b}{n}} = \{ gI \in G/I \mid gvg^{-1} \in \mathfrak{b} \}$$

Then

$H^*(\mathcal{B}_{\frac{mn+b}{n}})$ is a DAHA module

Oblomkov-Yun explain that

$$gr_* H^*(\mathcal{B}_{\frac{mn+b}{n}}) \text{ is a Rational Cherednik algebra module.}$$

So, by restriction, $gr_* H^*(\mathcal{B}_{\frac{mn+b}{n}})$ is an S_n-module.

Then

$$\sum_{i,j} t^j q^i \,\text{Frobenius characteristic}\left(gr_j H^i(\mathcal{B}_{\frac{m,n+b}{n}})\right) = Q_{mn+b,n}\mathbb{1}$$

(proof? reference?)

Hikita explained that the monomial expansion of the LHS is a direct consequence of Goresky-Kottwitz-Macpherson

$$\mathcal{B}_{\frac{mn+b}{n}} \cap IwI_0 = \mathbb{C}^{d(w)}$$

where $d(w) = \#\text{blue} - \#\text{red}$ between w_0 and w.